Devin Cela

1 (a) $\frac{\partial \tilde{\rho}}{\partial t} + \vec{v} \cdot \nabla \tilde{\rho} = -\rho_0 \vec{\nabla} \cdot \tilde{\vec{v}}$

$$\rho_0 \left(\frac{\partial \tilde{\vec{v}}}{\partial t} + \vec{v} \cdot \nabla \tilde{\vec{v}} \right) = -c_s^2 \nabla \tilde{\rho}$$

In our approximation neglecting spatial (or amplitude) variation,

$\tilde{\rho} = A e^{i\phi}$, $\tilde{\vec{v}} = \vec{B} e^{i\phi}$ (eikonal approximation, ϕ defined later)

$$iA \frac{\partial \phi}{\partial t} + \vec{v} \cdot (iA \vec{\nabla} \phi) = -\rho_0 (iB) \vec{\nabla} \phi \rightarrow iA \left[\frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi \right] = (-\rho_0 \vec{\nabla} \phi) iB$$

$$\rho_0 \left[\left(i \frac{\partial \phi}{\partial t} B \right) + \vec{v} \cdot (iB \vec{\nabla} \phi) \right] = -c_s^2 (iA \vec{\nabla} \phi) \rightarrow iB \left[\frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi \right] = \left(-\frac{c_s^2}{\rho_0} \vec{\nabla} \phi \right) iA$$

Multiply the two equations together:

$$\boxed{\left(\frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi \right)^2 = (\vec{\nabla} \phi)^2 c_s^2(x)}$$

$\phi = -\omega t + \vec{k} \cdot \vec{x}$. $\frac{\partial \phi}{\partial t} = -\omega$, $\vec{\nabla} \phi = \vec{k}$

$$(-\omega + \vec{v} \cdot \vec{k})^2 = k^2 c_s^2 \rightarrow \boxed{(\omega - \vec{k} \cdot \vec{v})^2 = k^2 c_s^2}$$

(b) Eikonal Equations: $\frac{d\vec{k}}{dt} = -\frac{\partial \omega}{\partial \vec{x}}$, $\frac{d\vec{x}}{dt} = \frac{\partial \omega}{\partial \vec{k}} = \vec{V}_{group}$ (derived in notes)

$\Phi = \int [\vec{k} \cdot d\vec{x} - \omega dt] \rightarrow$ total phase

$\delta \Phi = \delta \int [\vec{k} d\vec{x} - \omega dt] = 0 = \int \left\{ \delta \vec{k} \, d\vec{x} + \vec{k} \cdot \delta \, d\vec{x} - \frac{\partial \omega}{\partial \vec{k}} \delta \vec{k} + \frac{\partial \omega}{\partial \vec{x}} \cdot \delta \vec{x} \right\}$

$\delta \vec{x} = \delta \vec{k} = 0$, at end points, I.B.P.

$$\delta \Phi = \int \left\{ [\delta \vec{k} \cdot d\vec{x} - d\vec{k} \cdot \delta \vec{x}] - \left[\frac{\partial \omega}{\partial \vec{k}} \cdot \delta \vec{k} + \frac{\partial \omega}{\partial \vec{x}} \delta \vec{x} \right] dt \right\} \rightarrow \text{eikonal equations}$$

$$\frac{\partial \omega}{\partial \vec{k}} = \frac{d\vec{x}}{dt}, \quad -\frac{\partial \omega}{\partial \vec{x}} = \frac{d\vec{k}}{dt}$$

(b) $(\omega - \vec{k}\cdot\vec{V})^2 = (\pm kc)^2$

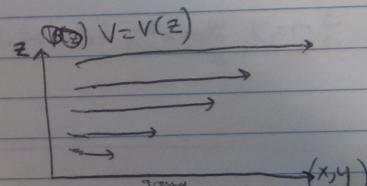
$$\partial\omega - \partial\vec{k}\cdot\vec{V} - \vec{k}\cdot\cancel{\partial\vec{V}} = \pm\partial k\, c\,\hat{k}$$

$$\frac{\partial \omega}{\partial \vec{k}} = \boxed{\vec{V} \pm c\hat{k} = \frac{\partial \vec{x}}{\partial t}}$$

$c(\vec{x})$ profile determines ray path

$$\omega = \vec{k}\cdot\vec{V} \pm kc \rightarrow -\frac{\partial \omega}{\partial x} = -\frac{\partial}{\partial x}\left(\vec{k}\cdot\vec{V} \pm kc\right) = \frac{d\vec{k}}{dt}$$

(c) $\frac{d\vec{k}}{dt} = -\frac{\partial}{\partial x}(\omega + \vec{k}\cdot\vec{V})$



(Vertically sheared horizontal flow)
(true of wind in reality)

Speed of the wave $\left(\frac{dk}{dt}\right)$ is now not only affected by its spatial variation in frequency ω, but by the variation $\left(\frac{\partial}{\partial x}\right)$ of the wave component in the direction of our flowing medium.

So our new term: $-\frac{\partial}{\partial z}(\vec{k}\cdot\vec{V}(z)) = -k_z \frac{dV}{dz}\hat{z}$, so $\frac{dk_z}{dt} \sim -k_z \frac{dV}{dz}$

On a windy day, $\frac{dV}{dz}$ is very high which affects the speed of your shouts.

(d) $$\boxed{\dot{x} = \frac{\partial \omega}{\partial \vec{k}} \leftrightarrow \dot{q} = \frac{\partial H}{\partial p} \;\Big|\; \dot{\vec{k}} = -\frac{\partial \omega}{\partial \vec{x}} \leftrightarrow \dot{p} = -\frac{\partial H}{\partial q}}$$ Analogous

In Hamiltonians, the phase space flow is incompressible $(\vec{\nabla}_\Gamma \cdot V_\Gamma = 0)$

As $V_\Gamma = V_\Gamma(\dot{q}, \dot{p})$ (flow), this means

$\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p} = 0$, or in our analogous case,

$$\frac{\partial}{\partial \vec{x}}\left(\frac{\partial \omega}{\partial \vec{k}}\right) + \frac{\partial}{\partial \vec{k}}\left(-\frac{\partial \omega}{\partial \vec{x}}\right) = \frac{\partial \omega}{\partial \vec{x}\partial \vec{k}} - \frac{\partial \omega}{\partial \vec{k}\partial \vec{x}} = 0 \checkmark$$

1. (d) (continued) (less cavalier version)

Showing that the wave equations with flow are ~~equivalent~~ Hamiltonian is equivalent to showing that the phase space flow is conserved: $\underline{\nabla}_r \cdot \underline{V}_r = 0$, where $\underline{V}_r = V_r\left(\frac{d\underline{x}}{dt}, \frac{d\underline{k}}{dt}\right)$

We have:

$$\underline{\nabla}_r \cdot \underline{V}_r = \frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} + \frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} = \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \omega_\ell}{\partial \underline{k}} + \underline{V}\right) + \frac{\partial}{\partial \underline{k}}\left(-\frac{\partial}{\partial \underline{x}}(\omega_\ell + \underline{k}\cdot\underline{V})\right)$$

(using $\omega_\ell = kc_s$)

$$\underline{\nabla}_r \cdot \underline{V}_r = \underbrace{\cancel{\frac{\partial^2 \omega_\ell}{\partial \underline{x}\partial \underline{k}}}}_{0} + \frac{\partial}{\partial \underline{x}} \cdot \underline{V} - \underbrace{\cancel{\frac{\partial^2 \omega_\ell}{\partial \underline{k}\partial \underline{x}}}}_{0} - \frac{\partial}{\partial \underline{k}}\left(\frac{\partial}{\partial \underline{x}}(\underline{k}\cdot\underline{V})\right)$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \underline{V} - \frac{\partial}{\partial \underline{k}}\left(\underline{k}\cdot\underline{\nabla}\,\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{k} + \underline{k}\times(\underline{\nabla}\times\underline{V}) + \underline{k}\times(\underline{\nabla}\times\underline{k})\right)$$

We know $\underline{k} = \underline{\nabla}\phi$, so $\underline{\nabla}\times\underline{k} = \underline{\nabla}\times(\nabla\phi) = 0$

We can also say: ~~[illegible]~~ $\frac{\partial}{\partial \underline{k}} \cdot \underline{V}\cdot\underline{\nabla}k = 0$

and $\frac{\partial}{\partial \underline{k}} \cdot \underline{k}\times(\underline{\nabla}\times\underline{V}) = 0$

Putting these together, we have

$$\underline{\nabla}_r \cdot \underline{V}_r = \frac{\partial}{\partial x} \cdot \underline{V} - \frac{\partial}{\partial \underline{k}}(\underline{k}\cdot\underline{\nabla}\,\underline{V})$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \underline{V} - \left[\frac{\partial}{\partial k_x}\underline{k}\cdot\underline{\nabla}V_x + \frac{\partial}{\partial k_y}\underline{k}\cdot\underline{\nabla}V_y + \frac{\partial}{\partial k_z}\underline{k}\cdot\underline{\nabla}V_z\right]$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \underline{V} - \underline{\nabla}\cdot\underline{V} = 0 \checkmark$$

PS IV (Presentation problem)

See page 187

Alexander Breindel

2.) So a boxed in spring has Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k_x x^2 - \frac{1}{2}k_y y^2 - \frac{1}{2}k_z z^2$$

$p_x = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$$

$p_x = \frac{\partial S}{\partial x}$

$p_y = \frac{\partial S}{\partial y}$ (eqn 35.14)

$p_z = \frac{\partial S}{\partial z}$

So the Hamilton-Jacobi equation is

$$\boxed{\frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right) + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2 + \frac{\partial S}{\partial t} = 0}$$

Since H is not explicitly dependent on time:

$$S(q, E, t) = W(q, E) - Et \qquad \text{Eqn (35.38)}$$

$$\frac{1}{2m}\left(\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right) + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2 = E$$

Since there are no cross terms assumes $W(q) =$

$W = W_1(x) + W_2(y) + W_3(z)$, so if $x_1 = x$, $x_2 = y$, $x_3 = z$, then

$$\sum_{i=0}^{3} \frac{1}{2m}\left(\frac{\partial W_i}{\partial x_i}\right)^2 + \frac{1}{2}k x_i^2 = E$$

$$\frac{1}{2m}\left(\frac{\partial W_1}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_2}{\partial y}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_3}{\partial z}\right)^2 + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$$

$$= E$$

$$\left.\begin{aligned}
\frac{1}{2m}\left(\frac{\partial W_1}{\partial x}\right)^2 + \frac{1}{2}k_x x^2 &= \Gamma_x \\
\frac{1}{2m}\left(\frac{\partial W_2}{\partial y}\right)^2 + \frac{1}{2}k_y y^2 &= \Gamma_y \\
\frac{1}{2m}\left(\frac{\partial W_3}{\partial z}\right)^2 + \frac{1}{2}k_z z^2 &= \Gamma_z
\end{aligned}\right\}$$ Constants of motion

$\sum \Gamma_i = E$, so Γ_i represents the amount of energy being dissipated in that direction

$$\left(\frac{\partial W_i}{\partial q_i}\right)^2 = 2m\left(\Gamma_i - \frac{1}{2}k_i q_i^2\right)$$

$$\frac{\partial W_i}{\partial q_i} = \sqrt{m\left(2\Gamma_i - \frac{1}{2}k_i q_i^2\right)}$$

$$W_i(q_i) = \int^{q} dq \left(m\left(2\Gamma_i - \frac{1}{2}k_i q_i^2\right)\right)^{1/2}$$

$$= \sqrt{\frac{m}{k_i}} \int^{q} dq \left(2\frac{\Gamma_i}{k_i} - q_i^2\right)^{1/2}$$

$$= \sqrt{\frac{m}{k}}\left(\frac{1}{2}\left(q\left(2\frac{\Gamma_i}{k_i} - q_i^2\right)^{1/2} + 2\frac{\Gamma_i}{k_i}\arcsin\left(\frac{q_i}{\sqrt{2\frac{\Gamma_i}{k_i}}}\right)\right)\right)$$

$$\boxed{\begin{aligned}
S = {} & \sqrt{\frac{m}{k_x}}\left(\frac{1}{2}\left(x\left(2\frac{\Gamma_x}{k_x} - x^2\right)^{1/2} + 2\frac{\Gamma_x}{k_x}\arcsin\left(\frac{x}{\sqrt{2\frac{\Gamma_x}{k_x}}}\right)\right)\right) \\
& + \sqrt{\frac{m}{k_y}}\left(\frac{1}{2}\left(y\left(2\frac{\Gamma_y}{k_y} - y^2\right)^{1/2} + 2\frac{\Gamma_y}{k_y}\arcsin\left(\frac{y}{\sqrt{2\frac{\Gamma_y}{k_y}}}\right)\right)\right) \\
& + \sqrt{\frac{m}{k_z}}\left(\frac{1}{2}\left(z\left(2\frac{\Gamma_z}{k_z} - z^2\right)^{1/2} + 2\frac{\Gamma_z}{k_z}\arcsin\left(\frac{z}{\sqrt{2\frac{\Gamma_z}{k_z}}}\right)\right)\right) \\
& - Et
\end{aligned}}$$

Gautam Reddy

3) $S = S(\rho, z, \phi ; E, \alpha, \beta)$

For $L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z)$

Since, $\frac{\partial L}{\partial t} = 0$,

$S = -Et + S_0(\rho, z, \phi ; E, \alpha, \beta)$

$$H = \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2m\rho^2} + \frac{p_z^2}{2m} + V(\rho, \phi, z)$$

$$= \frac{1}{2m}\left(\frac{\partial S_0}{\partial \rho}\right)^2 + \frac{1}{2m\rho^2}\left(\frac{\partial S_0}{\partial \phi}\right)^2 + \frac{1}{2m}\left(\frac{\partial S_0}{\partial z}\right)^2 + V(\rho, \phi, z)$$

For separability, we expect $V(\rho, \phi, z) = f_1(\rho) + \frac{f_2(\phi)}{\rho^2} + f_3(z)$.

$$\Rightarrow H = E = \frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial \rho}\right)^2 + 2mf_1\right] + \frac{1}{2m\rho^2}\left[\left(\frac{\partial S_0}{\partial \phi}\right)^2 + 2mf_2\right] + \frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial z}\right)^2 + 2mf_3\right]$$

Suppose we have a solution of the form, $S_0 = S_1(\rho ; E, \alpha, \beta) + S_2(\phi ; E, \alpha, \beta) + S_3(z ; E, \alpha, \beta)$

Plugging in,

$$E = \frac{1}{2m}\left[\left(\frac{\partial S_1}{\partial \rho}\right)^2 + 2mf_1\right] + \frac{1}{2m\rho^2}\left[\left(\frac{\partial S_2}{\partial \phi}\right)^2 + 2mf_2\right] + \frac{1}{2m}\left[\left(\frac{\partial S_3}{\partial z}\right)^2 + 2mf_3\right]$$

Each term in parentheses is a constant, say

$$\left(\frac{\partial S_2}{\partial \phi}\right)^2 + 2mf_2 = \alpha, \quad \left(\frac{\partial S_3}{\partial z}\right)^2 + 2mf_3 = \beta$$

$$\Rightarrow \left(\frac{\partial S_1}{\partial \rho}\right)^2 + 2mf_1 = 2mE - \frac{\alpha}{\rho^2} - \beta$$

We can now replace partial derivatives with ordinary derivatives and integrate.

$$\Rightarrow S_2 = \int \sqrt{\alpha - 2mf_2(\phi)}\, d\phi, \quad S_3 = \int \sqrt{\beta - 2mf_3(z)}\, dz.$$

$$S_1 = \int \sqrt{2mE - \frac{\alpha}{\rho^2} - \beta - 2mf_1(\rho)}\, d\rho$$

$$S = -Et + S_1 + S_2 + S_3$$

All information is contained in S.

4. Derive an expression for the relationship between the unit normal to an acoustic path and the profile of the index of refraction. Relate this result to the counterpart for particle motion, using the equation for a particle path.

To start, consider Maupertuis' Principle: $S_0 = \int_{q_1}^{q_2} \underline{p} \cdot d\underline{q}$

If we make the analogies: $S \longleftrightarrow \Phi$, $p \longleftrightarrow k$, $q \longleftrightarrow x$

We can write "Maupertuis' Principle for Rays" as

$$\Phi_0 = \int_{\underline{x}_1}^{x_2} \underline{k} \cdot d\underline{x} = \int_{\underline{x}_1}^{x_2} |\nabla \phi| \, ds$$

we can use the Eikonal Equation $(\nabla \phi)^2 = \frac{\omega^2}{c_0^2} n(\underline{x})^2 \rightarrow |\nabla \phi| = \frac{\omega}{c_0} n(\underline{x})$ to reduce this to

$$\Phi_0 = \frac{\omega}{c_0} \int_{\underline{x}_1}^{x_2} n(\underline{x}) \, ds$$

$\delta \Phi_0 = 0$ is clearly equivalent to Fermat's Principle

$$\delta \Phi_0 = \frac{\omega}{c_0} \int_{\underline{x}_1}^{x_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x} \, ds + n \, \delta(ds) \right\}$$

Note: $ds^2 = d\underline{x} \cdot d\underline{x} \rightarrow ds \, \delta(ds) = d\underline{x} \cdot \delta d\underline{x} \rightarrow \delta(ds) = \frac{d\underline{x}}{ds} \cdot d(\delta \underline{x})$

$$\rightarrow \delta \Phi_0 = \frac{\omega}{c_0} \int_{\underline{x}_1}^{x_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x} \, ds + n \frac{d\underline{x}}{ds} \cdot d(\delta \underline{x}) \right\}$$ integrate 2nd term by parts:

$$\rightarrow \delta \Phi_0 = \frac{\omega}{c_0} \left(\underbrace{n \frac{d\underline{x}}{ds} \cdot \delta \underline{x} \Big|_{\underline{x}_1}^{x_2}}_{\delta \underline{x} \text{ vanishes at endpoints:}} + \int_{\underline{x}_1}^{x_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x} \, ds - \frac{d}{ds}\left(n \frac{d\underline{x}}{ds}\right) \cdot \delta \underline{x} \, ds \right\} \right)$$

$$\rightarrow \delta \Phi_0 = \frac{\omega}{c_0} \int_{\underline{x}_1}^{x_2} \left\{ \frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n \frac{d\underline{x}}{ds}\right) \right\} \cdot \delta \underline{x} \, ds$$

$$\frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n \frac{d\underline{x}}{ds}\right) = 0$$ (Ray Equation)

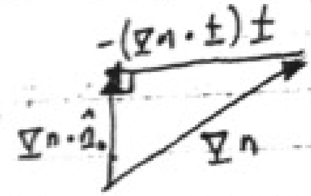
Expand ray equation: $n\frac{d^2\underline{x}}{ds^2} = \frac{\partial n}{\partial \underline{x}} - \left(\frac{\partial n}{\partial \underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds}$

$\rightarrow \frac{d^2\underline{x}}{ds^2} = \frac{1}{n}\frac{\partial n}{\partial \underline{x}} - \frac{1}{n}\left(\frac{\partial n}{\partial \underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds}$

$\frac{d\underline{x}}{ds} \equiv \underline{t}$ unit tangent to ray path

$\rightarrow \frac{d^2\underline{x}}{ds^2} = \frac{1}{n}\underline{\nabla} n - \frac{1}{n}(\underline{\nabla} n\cdot\underline{t})\,\underline{t}$

Geometry



$\hat{\underline{n}}_0 \equiv$ unit normal to ray path

$\rightarrow \frac{d^2\underline{x}}{ds^2} = \frac{1}{n}(\underline{\nabla} n\cdot\hat{\underline{n}}_0)\,\hat{\underline{n}}_0$

This is our relationship between the unit normal to the acoustic path and the profile of the index of refraction.

We can do the same thing for particles: Maupertuis' Principle gives:

$S_0 = \int\sum_j p_j\,dq_j \rightarrow \delta S_0 = \delta\int\sum_j p_j\,dq_j = 0$

$L = \frac{1}{2}\sum_{j,k} a_{j,k}(q)\,\dot{q}_j\dot{q}_k - U(q)\,,\quad p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_k a_{j,k}(q)\,\dot{q}_k$

$\rightarrow dS_0 = \sum_j p_j\,dq_j = \sum_{j,k} a_{j,k}(q)\,\dot{q}_k\,dq_j = \sum_{j,k} a_{j,k}(q)\frac{dq_k}{dt}\,dq_j$

For a conservative system, $\frac{1}{2}\sum_{j,k} a_{j,k}(q)\,\dot{q}_j\dot{q}_k + U(q) = E$ (constant)

$\rightarrow \frac{1}{2}\sum_{j,k} a_{j,k}(q)\frac{dq_j\,dq_k}{dt^2} = E-U \rightarrow dt = \left[\sum_{j,k} a_{j,k}(q)\,dq_j\,dq_k/2(E-U)\right]^{1/2}$

Plugging this in, we find $dS_0 = \int\left[2(E-U)\underbrace{\sum_{j,k} a_{j,k}(q)\,dq_j\,dq_k}_{ds^2}\right]^{1/2}$

For single particle, $a_{j,k}\rightarrow m \rightarrow \delta S_0 = \delta\int_{q_1}^{q_2}[2m(E-U)]^{1/2}\,ds$

$= \sqrt{2m}\,\delta\int_{q_1}^{q_2}[E-U]^{1/2}\,ds$

$\delta\int_{q_1}^{q_2}[E-U]^{1/2}\,ds = -\int\frac{\partial U}{\partial \underline{r}}\cdot\delta\underline{r}\,\frac{ds}{2(E-U)^{1/2}} + \int(E-U)^{1/2}\,d\delta s$

$ds^2 = d\underline{r}\cdot d\underline{r} \to ds\,d(\delta s) = d\underline{r}\cdot d(\delta\underline{r}) \to d(\delta s) = \frac{d\underline{r}}{ds}\cdot d(\delta\underline{r})$

$\to \delta\int (E-U)^{1/2}ds = -\int\left\{\frac{\partial U}{\partial\underline{r}}\cdot\frac{\delta\underline{r}}{2(E-U)^{1/2}}ds - (E-U)^{1/2}\frac{d\underline{r}}{ds}\cdot d(\delta\underline{r})\right\}$

integrate 2nd term of integrand by parts:

$\to \delta\int (E-U)^{1/2}ds = \underbrace{-(E-U)^{1/2}\frac{d\underline{r}}{ds}\cdot\delta\underline{r}\Big|_{r_1}^{r_2}}_{\delta\underline{r}\text{ vanishes at end points}} - \int_{r_1}^{r_2}\left\{\frac{\partial U}{\partial\underline{r}}\frac{1}{2(E-U)^{1/2}} + \frac{d}{ds}\left[(E-U)^{1/2}\frac{d\underline{r}}{ds}\right]\right\}\cdot\delta\underline{r}\,ds$

$\to \delta\int (E-U)^{1/2}ds = -\int_{r_1}^{r_2}\left\{\frac{\partial U}{\partial\underline{r}}\frac{1}{2(E-U)^{1/2}} + \frac{d}{ds}\left[(E-U)^{1/2}\frac{d\underline{r}}{ds}\right]\right\}\cdot\delta\underline{r}\,ds$

So for $\delta S_0 = 0$ we need $\frac{\partial U}{\partial\underline{r}}\frac{1}{2(E-U)^{1/2}} + \frac{d}{ds}\left[(E-U)^{1/2}\frac{d\underline{r}}{ds}\right] = 0$

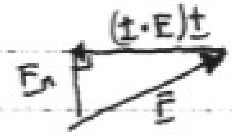
$\to 2(E-U)^{1/2}\frac{d}{ds}\left[(E-U)^{1/2}\frac{d\underline{r}}{ds}\right] = -\frac{\partial U}{\partial\underline{r}}$ particle path equation

Now define $\underline{t}\equiv\frac{d\underline{r}}{ds}$ "unit tangent to path", $\underline{F} = -\frac{\partial U}{\partial\underline{r}}$ "Force"

Expand particle path equation to get $\frac{d^2\underline{r}}{ds^2} = \frac{1}{2(E-U)}\left[-\frac{\partial U}{\partial\underline{r}} - \frac{d\underline{r}}{ds}\cdot\left(-\frac{\partial U}{\partial\underline{r}}\right)\frac{d\underline{r}}{ds}\right]$

$\to \frac{d^2\underline{r}}{ds^2} = \frac{1}{2(E-U)}\left[\underline{F} - (\underline{t}\cdot\underline{F})\underline{t}\right]$

Geometry



$\underline{F}_n \equiv$ Force normal to path $= \underline{F} - (\underline{t}\cdot\underline{F})\underline{t}$

$\to \frac{d^2\underline{r}}{ds^2} = \frac{1}{2(E-U)}\underline{F}_n$

This equation is analogous to $\frac{d^2\underline{x}}{ds^2} = \frac{1}{n}(\nabla n\cdot\hat{\underline{n}}_0)\hat{\underline{n}}_0$ for rays

again, we see $\nabla n \longleftrightarrow \underline{F}$

This is new $(E-U) \longleftrightarrow n$ (up to constant of proportion

KEVIN CREMIN

Problem 5:

For $n = n(x,y,z)$ & short wavelength, constant frequency

Consider $(\nabla\phi)^2 = \frac{\omega^2}{c_0^2} n(x,y,z)^2$

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2} n(x,y,z)^2$$

Now try seperation of variables $\phi(\underline{x}) = X(x) + Y(y) + Z(z)$

$$\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2} n^2$$

For this to solvable by this method, $n(x,y,z)^2$ must match the metric of rectangular coordinates. $\underline{n(x,y,z)^2 = a(x) + b(y) + c(z)}$

$$\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2}\left[a(x) + b(y) + c(z)\right]$$

Gather terms together

$$\underbrace{\left\{\left(\frac{\partial X}{\partial x}\right)^2 - \frac{\omega^2}{c_0^2}a(x)\right\}}_{=k_x} + \underbrace{\left\{\left(\frac{\partial Y}{\partial y}\right)^2 - \frac{\omega^2}{c_0^2}b(y)\right\}}_{=k_y} + \underbrace{\left\{\left(\frac{\partial Z}{\partial z}\right)^2 - \frac{\omega^2}{c_0^2}c(z)\right\}}_{=k_z = -(k_x+k_y)} = 0$$

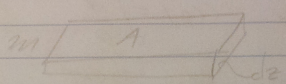
$$\frac{\partial X}{\partial x} = \left[k_x + \frac{\omega^2}{c_0^2}a(x)\right]^{1/2}$$

$$X = \pm\int dx\left[k_x + \frac{\omega^2}{c_0^2}a(x)\right]^{1/2} \quad Y = \pm\int dy\left[k_y + \frac{\omega^2}{c_0^2}b(y)\right]^{1/2} \quad Z = \pm\int dz\left[-(k_x+k_y) + \frac{\omega^2}{c_0^2}c(z)\right]^{1/2}$$

$$\phi(\underline{x}) = X + Y + Z$$

$$\boxed{\phi(\underline{x}) = \pm\int dx\left[k_x + \frac{\omega^2}{c_0^2}a(x)\right]^{1/2} \pm \int dy\left[k_y + \frac{\omega^2}{c_0^2}by\right]^{1/2} \pm \int dz\left[-(k_x+k_y) + \frac{\omega^2}{c_0^2}c(z)\right]^{1/2}}$$

(6) First find the density change with altitude; consider a sheet of air of mass dm, in hydrostatic equilibrium.



Then,
$$dF_{grav} = dP\,A$$
$$-g\,dm = g\rho A\,dz = A\,dP$$

We can get ρ from the ideal gas law:
$$PV = nk_BT \rightarrow \rho \sim \frac{n}{V} = \frac{PM}{k_BT}$$
Then we have
$$\int \frac{-g}{k_BT}\,dz = \int dP/P$$
$$-\frac{gz}{k_BT} = \ln P + P_0$$
$$\Rightarrow \boxed{P(z) = P_0 e^{-Mgz/k_BT}}$$
in particular, $\boxed{\rho(z) = \frac{PM}{k_BT} = \frac{MP_0}{k_BT} e^{-Mgz/k_BT}}$

Now use the helmholtz eqn. with ansatz $\psi = A(z)e^{i\phi(z)}$.
$$\nabla^2\psi = -\frac{\omega^2}{c(z)^2}\psi$$
$$\Rightarrow \frac{-\omega^2}{c^2} = \nabla^2 A + 2i\nabla A\cdot\nabla\phi + i\nabla^2\phi A - (\nabla\phi)^2 A$$
Taking the real part:
$$\frac{-\omega^2}{c(z)^2} = \nabla^2 A + (\nabla\phi)^2 A$$
Now, $\phi = \int k\,dz - \omega t \Rightarrow \nabla\phi = \frac{\partial\phi}{\partial z} = k$ ($\neq k_B$!) OVER $\rightarrow$

$$\frac{-\omega^2}{c(z)^2} = \nabla^2 A + Ak$$

Now, we know $c(z) = \sqrt{\frac{\gamma}{\rho}}$ (Newton-Laplace eqn.)

so $\frac{1}{c(z)^2} = \frac{\rho}{\gamma} = \frac{\rho_0 M}{\gamma k_B T} e^{-Mgz/k_B T}$

& we have

$$\frac{-\omega_0 \rho_0 M}{\gamma k_B T} e^{-Mgz/k_B T} = \frac{d^2 A}{dz^2} + Ak$$

This has solution:

$$\boxed{A(z) = \frac{-\omega_0^2 \rho_0 M (k_B T)}{\gamma} \frac{e^{-Mgz/(k_B T)}}{k(k_B T) + (Mg)^2}}$$ + oscilla term

As $g \to 0$ this reduces to a constant, as we expects.

#17 Consider $\frac{d^2\psi}{dx^2} + \frac{Q(x)}{\varepsilon^2}\psi = 0$

a.) Guess $\psi = e^{i/\varepsilon \sum_{n=0}^{\infty} \varepsilon^n \Phi_n(x)}$

if ε is small, look at Φ_0, Φ_1, & Φ_2

$$\psi \simeq e^{i/\varepsilon[\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2]}$$

$$\partial\psi/\partial x = i/\varepsilon\left[\Phi_0' + \varepsilon\Phi_1' + \varepsilon^2\Phi_2'\right] e^{i/\varepsilon[\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2]}$$

$$\partial^2\psi/\partial x^2 = \left(\frac{-1}{\varepsilon^2}\left[\Phi_0' + \varepsilon\Phi_1' + \varepsilon^2\Phi_2'\right]^2 + i/\varepsilon\left[\Phi_0'' + \varepsilon\Phi_1'' + \varepsilon^2\Phi_2''\right]\right) e^{i/\varepsilon[\dots]}$$

Plug into Helmholtz

Highest order $1/\varepsilon^2$ terms:

$$\boxed{-(\Phi_0')^2 + Q = 0}$$

2nd highest order $1/\varepsilon$ terms:

$$\boxed{-2\Phi_0'\Phi_1' + i\Phi_0'' = 0}$$

3rd highest order terms

$$\boxed{-2\Phi_0'\Phi_2' + -(\Phi_1')^2 + i\Phi_1'' = 0}$$

b.) The 1st equation is the eikonal eq.

$$(\Phi_0')^2 = Q(x) \qquad \text{compare to} \quad (\nabla\Phi)^2 = \omega^2/c^2$$

c.) $$\boxed{\Phi_0 = \pm\int\sqrt{Q(x)}\,dx}$$

$$2\Phi_0'\Phi_1' = i\Phi_0''$$

$$\Phi_1' = \frac{i}{2}\frac{\Phi_0''}{\Phi_0'} \Rightarrow \int\Phi_1' = \frac{i}{2}\int\frac{\Phi_0''}{\Phi_0'}$$

$$\Phi_1 = i/2\ln(\Phi_0') + C$$

$$\phi_1 = \frac{i}{2} \ln\left(\sqrt{Q(x)}\right) + c$$

$$\boxed{\phi_1 = i \ln\left(Q^{1/4}\right) + c'}$$

$$\psi \approx e^{\frac{i}{\varepsilon}\left[\pm \int \sqrt{Q(x)}\, dx\right]} e^{\frac{i}{\varepsilon}\left[\varepsilon\left(i \ln\left(Q^{1/4}\right) + c'\right)\right]}$$

$$\boxed{\psi \approx \frac{A e^{\pm \frac{i}{\varepsilon}\left(\int \sqrt{Q(x)}\, dx\right)}}{Q^{1/4}}}$$

$A =$ some const.

d.) This is valid for small ε and $Q(x)$ such that $\sum \varepsilon^n \phi_n(x)$ converges